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LETTER TO THE EDITOR

Exact critical exponents for two-dimensional dense polymers

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Abstract. Using conformal invariance and Coulomb gas results, we give the exact value in two dimensions of the η exponent of L dense polymers, attached by their extremities: $\eta_L = (L^2 - 4)/8$. The value in two dimensions of the γ exponent of a dense branched polymer of fixed topology with n_L L -leg vertices, $L \geq 1$ is then deduced to be

$$\gamma = \sum_{L \geq 1} n_L (2 - L)(L + 18)/32.$$

These values correspond to a conformally invariant theory with central charge $C = -2$.

Recently (Duplantier 1986a, b, hereafter referred to as I and II), we have proposed an exact formula for the critical exponent γ associated with a self-avoiding polymer network in two dimensions, for any fixed topology of the network (I), and we have given (II) exact contact critical exponents for a single self-avoiding walk. To do this, we used, besides renormalisation theory, information obtained from conformal invariance theory in two dimensions (Dotsenko and Fateev 1984) and numerical studies on strips (Saleur 1986a).

The predictions were in quite good agreement with existing numerical simulations for star polymers (Wilkinson *et al* 1986), H-comb polymers (Gaunt *et al* 1986) and contact exponents (Redner 1980 and references therein). The results applied to dilute polymer structures floating in a good solvent, and swollen by excluded-volume effects. In this letter, our aim is to derive similar results for *dense* self-avoiding polymer networks in two dimensions. By dense, we mean that the polymer fills, even in the thermodynamic limit, a finite fraction of the embedding lattice. To do this, we shall use conformal invariance applied to the low temperature phase (i.e. high fugacity in polymer language) of the $O(n)$ n -vector model, in a vanishing small magnetic field, and in the limit $n = 0$. As is well known, the low fugacity or high temperature phase corresponds in contrast to polymers in a dilute solution (see, e.g., de Gennes 1979) or, in a non-zero magnetic field, to a semi-dilute solution (des Cloizeaux 1975). Dense polymers have been considered in various studies (Parisi and Sourlas 1980, Gujrati 1985, Gaspari and Rudnick 1986 and references therein). Dense polymers may also have some connections with Hamiltonian walks (Kasteleyn 1963, Orland *et al* 1985 and references therein), but the latter have certainly much less universal properties (see, e.g., Gordon *et al* 1976) than those of dense polymers, associated with the $O(n = 0)$ model.

To obtain our results for branched polymers of arbitrary topology, we first study (I) one particular 'fuseaux' configuration of L polymer chains attached to each other by their extremities (figure 1). The *polydisperse* partition function of this network with fixed extremities at 0 and r on a lattice is

$$Z_L(r, 0, \beta) = \sum_{\mathcal{C}(0,r)} \beta^{s(\mathcal{C})} \quad (1)$$

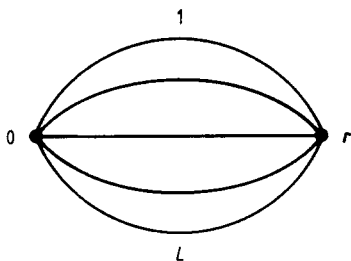


Figure 1. 'Fuseaux' network made of L chains attached by their extremities.

where the sum is taken over all self-avoiding configurations $\mathcal{C}(0, r)$ connecting 0 and r , of total length $S(\mathcal{C})$, β being the fugacity. The critical fugacity is $\beta_c = \mu^{-1}$, where μ is the self-avoiding walk effective connectivity constant. For $\beta > \beta_c$, the polymer will fill a finite fraction of the lattice. At the critical point

$$Z_L(r, 0, \beta_c) \sim r^{-2x_L}. \quad (2)$$

The critical exponent x_L depends on the side from which β_c is approached: from the high temperature phase (low fugacity $\beta < \beta_c$) or from the low temperature phase (high fugacity $\beta > \beta_c$). In terms of conformal invariance theory (Belavin *et al* 1984, Dotsenko and Fateev 1984), Z_L can be written as

$$Z_L(r, 0, \beta_c) / Z(\beta_c) = \langle \phi_L(r) \phi_L(0) \rangle \quad (3)$$

where the ϕ_L are conformal operators whose conformal dimensions are x_L . The conformal theory to which they belong will be *different* for $\beta \rightarrow \beta_c^-$ and $\beta \rightarrow \beta_c^+$. $Z(\beta_c)$ in (3) represents the partition function of the $O(n)$ model, $n \rightarrow 0$ (its value also depends on the side from which β_c is approached). In the dense phase, we find the new value of x_L in two dimensions:

$$x_L = (L^2 - 4) / 16. \quad (4)$$

This is done as follows. In the dilute phase ($\beta < \beta_c$), the value of x_L in two dimensions has been identified (Saleur 1986a, b) from numerical transfer matrix calculations on strips:

$$x_L = (9L^2 - 4) / 48 \quad (5)$$

corresponding to the Kac formula (Kac 1979), for $L = 2p - 1$, $x_L = 2h_{p+1/2, 3/2}$ and for $L = 2p$, $x_L = 2h_{p+2, 3}$, for a conformally invariant theory with central charge $C = 0$. To obtain x_L (4), we first note that the other x_L (5) (for L odd), coincides for $n = 0$ with the magnetic exponents X_H of the $O(n)$ model given by Dotsenko and Fateev (1984):

$$\begin{aligned} x_{L=2p-1} &= X_{H_p} = [L^2 - (2-t)^2] / 4t \\ &= L^2 g / 8 - (g-1)^2 / 2g \end{aligned} \quad (6)$$

where t or g are variables parametrising the $O(n)$ model by (Cardy and Hamber 1980, Nienhuis 1982, 1984)

$$n = -2 \cos 2\pi/t = -2 \cos \pi g. \quad (7)$$

Here $g = 2/t$ is the renormalised coupling constant appearing in Nienhuis' study of the $O(n)$ model by the Coulomb gas method. According to Nienhuis (1982, 1984),

the *high* temperature phase of the $O(n)$ model is given by the analytic determination of g which satisfies (7) and $1 < g < 2$, whilst the *low* temperature phase corresponds to

$$0 < g < 1. \tag{8}$$

Hence for $n = 0$, we find for $\beta < \beta_c$

$$g = \frac{3}{2} \quad t = \frac{4}{3} \tag{9a}$$

whilst for $\beta > \beta_c$

$$g = \frac{1}{2} \quad t = 4. \tag{9b}$$

Inserting (9a) into (6) recovers (5), as expected, whilst (9b) gives our new result (4) for dense polymers, as expected. Of interest also is the value of the central charge C corresponding to the *dense* phase of the polymers. It can be found by using the correspondence between parameter t (7) and central charge C conjectured by Dotsenko and Fateev (1984) and established by Blöte *et al* (1986) for $O(n)$ and Potts models. Using (9b), we find for dense polymers the new result

$$C = -2 \tag{10}$$

(whilst for dilute polymers $C = 0$). Using the Kac formula (1979), we further check that the critical exponents x_L (4) are in the conformal table of $m = 1$, parametrising $C = 1 - 6/m(m + 1)$:

$$x_L = 2h_{p/2,1/2} \quad L = 2p - 1$$

$$x_L = 2h_{(p+1)/2,1} \quad L = 2p.$$

It is interesting to note that the Hamiltonian walk problem on a periodic Manhattan lattice also has $C = -2$. This can be seen by expanding the free energy of Kasteleyn's solution (1963) along one infinite direction on a strip of width w . Then identifying the universal finite-size correction term $C\pi/6w^2$, given by Blöte *et al* (1986) and Affleck (1986), one finds $C = -2$. For $L = 1$ (a single polymer), equation (4) gives an η exponent $\eta_1 \equiv 2x_1 = -\frac{3}{8}$, whilst for $L \geq 2$ the η exponents $\eta_L = 2x_L$ are null or positive. $\eta_1 < 0$ corresponds to an effective repulsion between the extremities of a dense polymer in 2D (Gaspari and Rudnick 1986). η_1 has been studied numerically on strips by Saleur (1986c), who also observed the value $C = -2$. For $L = 2$, the thermal exponent x_2 is $x_2 = 2 - 1/\nu$, and hence $x_2 = 0$ gives

$$\nu = \frac{1}{2} \tag{11}$$

which corresponds to a close-packed polymer in 2D: $\nu = 1/d$, as it must.

Let us consider now a dense polymer network (or branched polymer) of fixed topology (figure 2), made of a number \mathcal{N} of large chains, attached together by n_L L -leg vertices, $L \geq 1$ ($L = 1$: dangling ends). The chains all have the same large length S . Then the partition function for the network scales as

$$\mathcal{Z}_{\mathcal{G}} \sim S^{\gamma_{\mathcal{G}} - 1} \tag{12}$$

where $\gamma_{\mathcal{G}}$ is a critical exponent depending on the topology of \mathcal{G} . Knowing the basic exponents x_L , we are able (I) to calculate $\gamma_{\mathcal{G}}$. We find (in two dimensions)

$$\gamma_{\mathcal{G}} - 1 = \nu \left(- \sum_{L \geq 1} n_L x_L + 2(\mathcal{V} - 1) \right) - \mathcal{N} \tag{13}$$

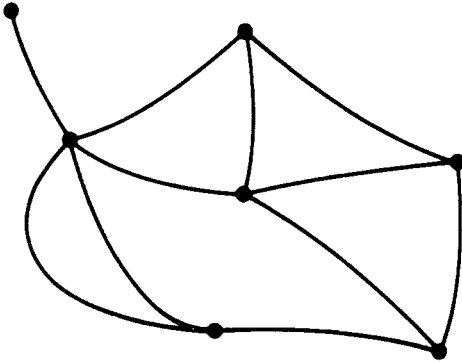


Figure 2. A network made of $\mathcal{N} = 11$ chains, $\mathcal{L} = 5$ loops and vertices with $L = 1, \dots, 5$ legs, with $n_{11}, n_3 = 4, n_4 = 1, n_5 = 1$.

where $\mathcal{V} = \sum_{L \geq 1} n_L$ is the total number of vertices of the polymer, and \mathcal{N} is $2\mathcal{N} = \sum_{L \geq 1} Ln_L$. Applying results (4) and (11), we find for a dense network the *new infinite set of critical exponents* in two dimensions

$$\gamma_{\mathcal{G}} = \sum_{L \geq 1} n_L (2 - L)(L + 18) / 32. \quad (14)$$

Note that $L = 2$ does not contribute, as it must (trivial point insertions). For a single polymer chain ($n_1 = 2, n_{L \neq 1} = 0$) the resulting exponent γ is

$$\gamma = \frac{19}{16}. \quad (15)$$

For a dense L -star polymer, corresponding to $n_L = 1, n_1 = L$, we find the new result

$$\gamma_L = (-L^2 + 3L + 36) / 32. \quad (16)$$

In I instead of (13), we used a similar equation in terms of basic *vertex* critical exponents $\hat{\sigma}_L$, which read in terms of x_L in d -dimensional space

$$\hat{\sigma}_L = -\nu x_L + (\nu d - 1)L / 2. \quad (17)$$

Thus, for *dense* polymers in $d = 2$, (4) and (11) give the set of exact values

$$\hat{\sigma}_L = -(L^2 - 4) / 32. \quad (18)$$

A word of caution is necessary here. All along we have assumed that the general universal features of the $O(n)$ model will lead, when applied in the ordered phase, to a universal behaviour for $n < 1$. Thus we expect our result (equations (12) and (14)) for $n = 0$ to describe configurations of dense polymers which *nearly* 'fill' the space. There are subtleties due to boundary conditions which may have influence on such a dense phase. However, we expect that they do not affect the universal behaviour (12) and (14).

Let us finally consider the *contact* exponents of a *single* polymer chain, defined in II. We select configuration of a SAW where *fixed* points of the SAW are very close together (figure 3). By coalescing these points, one obtains a graph \mathcal{G} of contacts (figure 3). Then the probability $P_{\mathcal{G}}$ of making this set of contacts scales like (see II)

$$P_{\mathcal{G}} = S^{-Y_{\mathcal{G}}} \quad (19)$$

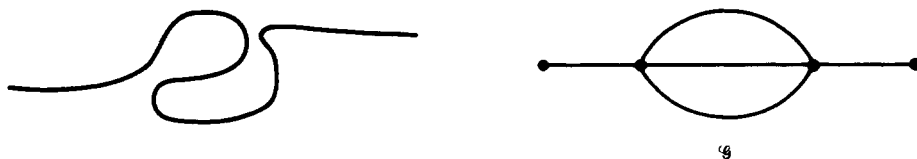


Figure 3. A set of contacts inside a single polymer and the corresponding polymer network or graph \mathcal{G} .

where $Y_{\mathcal{G}}$ is the probability contact exponent

$$Y_{\mathcal{G}} = \gamma - \gamma_{\mathcal{G}} \quad (20)$$

Using then (14) and (15), we find for dense polymers

$$Y_{\mathcal{G}} = \frac{19}{16} + \sum_{L \geq 1} n_L (L-2)(L+18)/32. \quad (21)$$

Accordingly, the contact exponent $\theta_{\mathcal{G}}$ with which the different points of contact approach together in space (see II) satisfies $Y_{\mathcal{G}} = \nu(d\mathcal{L} + \theta_{\mathcal{G}})$, where \mathcal{L} is the number of loops of graph \mathcal{G} : $\mathcal{L} = \sum_{L \geq 1} n_L (\frac{1}{2}L - 1) + 1$. Then $\theta_{\mathcal{G}}$ for a dense single polymer in two dimensions is

$$\theta_{\mathcal{G}} = \frac{3}{8} + \frac{1}{16} \sum_{L \geq 1} n_L (L^2 - 4). \quad (22)$$

For instance, we find for the known θ_1 , θ_2 exponents (des Cloizeaux 1980, Redner 1980), which of them corresponds to the contact of one extremity of the chain inside the latter (θ_1), and which to the contact of two interior points (θ_2). θ_1 corresponds to a graph of contact \mathcal{G}_1 : $n_1 = 1$, $n_3 = 1$, θ_2 to \mathcal{G}_2 : $n_1 = 2$, $n_4 = 1$. Hence

$$\theta_1 = \frac{1}{2} \quad \theta_2 = \frac{3}{4}. \quad (23)$$

As expected, they are smaller than the dilute exponents which we have given in (II): $\theta_1 = \frac{5}{6}$, $\theta_2 = \frac{19}{12}$: excluded-volume effects are screened in the dense phase.

In summary, we have used results of conformal invariance in two dimensions and the analytic continuation to the ordered phase of the $n = 0$ vector model for obtaining new critical exponents of dense self-avoiding polymers in two dimensions. We have shown that these exponents correspond to a conformally invariant theory with central charge $C = -2$. We have proposed an exact formula for the exponent γ of a general dense polymer network in two dimensions, as a function of its topology.

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